Equivalence of partition functions for noncommutative $\mathrm{U}(1)$ gauge theory and its dual in phase space

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Abstract: Equivalence of partition functions for $U(1)$ gauge theory and its dual in appropriate phase spaces is established in terms of constrained hamiltonian formalism of their parent action. Relations between the electric–magnetic duality transformation and the (S) duality transformation which inverts the strong coupling domains to the weak coupling domains of noncommutative $U(1)$ gauge theory are discussed in terms of the lagrangian and the hamiltonian densities. The approach presented for the commutative case is utilized to demonstrate that noncommutative $U(1)$ gauge theory and its dual possess the same partition function in their phase spaces at the first order in the noncommutativity parameter θ.

Keywords: Field theory, noncommutative field theory, duality.

Contents

1. Introduction

Maxwell equations in vacuum are electric–magnetic duality invariant. Similarly one can formulate a duality transformation of $U(1)$ gauge theory action: This (S) duality inverts weak coupling constant regions into strong coupling constant regions. A parent action [1] is defined in terms of the dual gauge field A_D and the antisymmetric second rank tensor F. When it is employed in the path integral, if one integrates over A_D the partition function of the ordinary $U(1)$ theory results. Instead of A_D one can integrate F which yields the partition function of the dual $U(1)$ theory. Thus, one can easily show equivalence of partition functions for the $U(1)$ and its dual theory, up to a normalization constant. On the other hand hamiltonian description of these theories are shown to be connected by a canonical transformation and as a consequence it followed that the partition functions in their phase spaces are the same[2]. We demonstrate that this equivalence can directly be obtained in terms of hamiltonian formulation of the parent action: Utilizing constraints one can integrate the desired phase space variables obtaining either the partition function of $U(1)$ gauge theory or the partition function of its dual theory in appropriate phase spaces.

In terms of fields taking values in a noncommutative space one can introduce a noncommutative $U(1)$ gauge theory. However, these noncommuting fields can be mapped into ordinary fields utilizing the Seiberg–Witten map[3]. Then, a dual noncommutative $U(1)$ action can be obtained analogous to the commutative case, by introducing a parent action[4]. When the initial $U(1)$ theory possesses a spatial noncommutativity the dual one is also noncommutative $U(1)$ gauge theory whose time coordinate is noncommuting with spatial coordinates. Hamiltonian formulation

of the latter theory, which is suitable to study the noncommutative D3–brane, was presented in [5].

Although electric–magnetic duality transformation is an invariance of Maxwell equations in vacuum, it is known that it maps the lagrangian density to itself up to an overall minus sign and keeps intact the hamiltonian density of $U(1)$ gauge theory. Electric–magnetic duality transformation of the equations of motion of noncommutative $U(1)$ theory is studied in [6]. In spite of that, we would like to understand the relation between electric–magnetic duality and the (S) duality inverting strong and weak coupling regimes. Hence, we discuss relations of the electric–magnetic duality with the dual description of the noncommutative gauge theory utilizing the lagrangian and the hamiltonian densities. We only deal with the first order approximation in the noncommutativity parameter θ.

For $U(1)$ gauge theory the parent action can be used in the related path integrals to derive the duality symmetry between the original and the dual theories. But, for the noncommutative theory one should employ the equations of motion derived from the parent action, to obtain the dual noncommutative $U(1)$ theory action[4]. The (S) duality symmetry of the noncommutative $U(1)$ theory was not established and relation between their partition functions was unknown. We show that partition functions for the noncommutative $U(1)$ theory with spatial noncommutativity and its dual whose time coordinate is effectively noncommuting with spatial coordinates, are equivalent in the appropriate phase spaces. To achieve this we follow the approach presented for the commutative gauge theory.

In Section 2 we first present the constrained Hamiltonian structure of the parent action of Maxwell theory. The partition function of the parent action in phase space is written. We show that by integrating over the appropriate fields either the partition function of $U(1)$ theory in its phase space or the partition function of its dual theory in dual phase space results.

Relations between the electric–magnetic duality transformations and the (S) dual actions of noncommutative $U(1)$ gauge theory are discussed in terms of configuration space fields as well as in terms of phase space fields in Section 3.

In Section 4, guided by the approach of Section 2, the path integral of parent action for the noncommutative theory in phase space is studied. We demonstrate equivalence of partition functions for spatially noncommutative $U(1)$ gauge theory and its dual being effectively space–time noncommutative $U(1)$ gauge theory with an inverted coupling constant, at the first order in θ.

2. Partition functions for $U(1)$ gauge theory and its dual

In Minkowski space–time with the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, $U(1)$ gauge theory

and its dual can be extracted from the parent action

$$S_P = -\int d^4x \left(\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\mu A_D^\nu F^{\rho\sigma} \right). \tag{2.1}$$

Here, $F_{\mu\nu}$ are not related to gauge fields, they are the basic variable fields. Let us introduce the canonical momenta $P_{\mu\nu}$ and $P_{D\mu}$ corresponding to $F_{\mu\nu}$ and $A_{D\mu}$. Definitions of the canonical momenta P_D^μ, $P_{\mu\nu}$, yield the weakly vanishing primary constraints

$$\Phi^1_{\mu\nu} \equiv P_{\mu\nu} \approx 0, \tag{2.2}$$

$$\xi^1 \equiv P_{D0} \approx 0, \tag{2.3}$$

$$\chi^2_i \equiv P_{Di} + \frac{1}{2} \epsilon_{ijk} F^{jk} \approx 0. \tag{2.4}$$

Canonical hamiltonian associated with the parent action (2.1) is

$$H_{PC} = \int d^3x \left[\frac{1}{2g^2} F^{0i} F_{0i} + \frac{1}{4g^2} F^{ij} F_{ij} - \frac{1}{2} \epsilon_{ijk} \partial^i A_D{}^0 F^{jk} + \epsilon_{ijk} \partial^i A_D{}^j F^{0k} \right]. \tag{2.5}$$

Consistency of the primary constraints (2.2)–(2.4) with the equations of motion resulting from (2.5) gives rise to the secondary constraints

$$\Phi^3 \equiv \{P_{D0}, H_{PC}\} = \epsilon_{ijk} \partial^i F^{jk} \approx 0, \tag{2.6}$$

$$\chi^4_i \equiv \{P_{0i}, H_{PC}\} = F_{0i} + g^2 \epsilon_{ijk} \partial^j A_D{}^k \approx 0. \tag{2.7}$$

Let us find out the number of physical phase space fields: The constraint (2.3) is obviously first class. Besides it, the linear combination

$$\xi^2 \equiv \partial_i \chi^2_i - \frac{1}{2} \Phi^3 = \partial_i P_{Di} \approx 0, \tag{2.8}$$

is also a first class constraint. A vector can be completely described by giving its divergence and rotation (up to a boundary condition). (2.8) is derived taking divergence of χ^2_i, so that, there are still two linearly independent second class constraints following from the curl of χ^2_i. Obviously, the constraints Φ^1, Φ^3, χ^4 are all second class and linearly independent. Therefore, the number of physical phase space fields is four.

To deal with path integrals, we choose the gauge fixing (subsidiary) conditions

$$\Lambda^1 = A_{D0} \approx 0, \ \Lambda^2 = \partial_i A_{Di} \approx 0 \tag{2.9}$$

for the first class constraints (2.3) and (2.8). The linearly independent second class constraints resulting from the curl of χ^2_i can be taken as

$$\Phi^2_n \equiv C^i_n \chi^2_i \equiv K^i_n \epsilon_{ijk} \partial_j \chi^2_k \approx 0, \tag{2.10}$$

where $n=1,2$, and K_n^i are some constants which should be chosen in accordance with solutions of the other constraints when they vanish strongly. Instead of dealing with χ_i^4 we introduce another set of linearly independent second class constraints:

$$\Phi_n^4 \equiv M_n^i \chi_i^4 \equiv L_n^i \epsilon_{ijk} \partial_j \chi_k^4 \approx 0, \quad \Phi_3^4 \equiv \partial^i F_{0i} \approx 0. \tag{2.11}$$

L_n^i are some constants. As we will see, explicit forms of K_n^i and L_n^i play no role in our calculations.

Partition function associated with the hamiltonian (2.5) in the total phase space is

$$Z = \int DA_D DF DP_D DP \; \Delta \exp\left\{ i \int d^3x \left[P_{D\mu} \dot{A}_D^\mu + P_{\mu\nu} \dot{F}^{\mu\nu} - H_{PC} \right] \right\}. \tag{2.12}$$

We suppressed the indices of the integration variables and the measure Δ is defined[7],[8] as

$$\Delta = \det\{\xi^\alpha, \Lambda^\beta\} \det{}^{1/2}\{\Phi^a, \Phi^b\} \prod_{\sigma=1}^{2} \delta(\xi^\sigma)\delta(\Lambda^\sigma) \prod_{c=1}^{4} \delta(\Phi^c). \tag{2.13}$$

The determinant related to first class constraints and their subsidiary conditions is

$$\det\{\xi^\alpha, \Lambda^\beta\} = \det \partial_i \partial^i \equiv \det(\partial^2).$$

The determinant due to the second class constraints can be calculated as

$$\det{}^{1/2}\{\Phi^a, \Phi^b\} = \det\left(\epsilon_{ijk} \partial^i C_1^j C_2^k\right) \det\left(\epsilon_{ijk} \partial^i M_1^j M_2^k\right), \tag{2.14}$$

where the linear differential operators C_n^i and M_n^i are defined in (2.10) and (2.11). Here, the determinants of these linear operators should be interpreted as multiplication of their eigenvalues.

Performing functional integrations over the variables $F^{\mu\nu}$,$P_{\mu\nu}$ and A_D^0, P_D^0 we obtain

$$\begin{aligned} Z = & \int D\mathbf{A}_D D\mathbf{P}_D \delta(\partial \cdot \mathbf{P}_D)\delta(\partial \cdot \mathbf{A}_D) \det(\partial^2) \\ & \exp\left\{ i \int d^3x \left[P_{Di} \dot{A}_D^i - \frac{1}{2g^2} P_{Di} P_D^i - \frac{g^2}{4} F_D^{ij} F_{Dij} \right] \right\}. \end{aligned} \tag{2.15}$$

Here, the factor $\det^{1/2}\{\Phi^a, \Phi^b\}$ is canceled with the determinant arising from the Dirac delta functions $\delta(\Phi^a)$ when we use them to express $F_{\mu\nu}$, $P_{\mu\nu}$ in terms of the "physical" fields $\mathbf{A}_D$, $\mathbf{P}_D$. Although here this can be observed by direct calculation[1], it is true in general when one gets rid of second class constraints by imposing them strongly and deal with reduced phase space path integrals[8].

[1]To obtain (2.15) we do not need to deal with the set (2.11). It is easier to employ (2.7) with an appropriate determinant.

Now, in (2.12) we would like to perform integrations over the dual fields $A_{D\mu},\ P_{D\mu}$ and the momenta $P_{\mu\nu}$. Vanishing of the constraint (2.6) strongly, i.e. $\Phi^3=0$, dictates that

$$F_{ij}=\partial_i A_j-\partial_j A_i. \tag{2.16}$$

Being a second class constraint $\Phi^3=0$ should eliminate one phase space variable. However, the number of independent components of F_{ij} and A_i are the same. So that, solving $\Phi^3=0$ as (2.16) and dealing with A_i instead of F_{ij}, has to be accompanied with a condition on A_i. The constraint (2.4) involves only curl of A_i, therefore, $\Phi_n^2=0$ give information only about the two components of A_i. In order to describe A_i completely one needs to furnish its divergence. Thus, we choose as the missing condition

$$\partial_i A^i=0. \tag{2.17}$$

After performing $A_{D\mu}, P_{D\mu}$ and $P_{\mu\nu}$ integrations in (2.12) we obtain

$$\begin{aligned}Z=\det g^{-4}\int D\mathbf{A}DF_{0j}\det(\partial^2)\delta(\partial^l F_{0l})\delta(\partial\cdot\mathbf{A})\\ \exp\left\{i\int d^3x[-\frac{1}{g^2}F_{0i}\dot{A}^i+\frac{1}{2g^2}F^{0i}F_{0i}-\frac{1}{4g^2}F^{ij}F_{ij}]\right\}.\end{aligned} \tag{2.18}$$

We used the fact that expressing A_{Di} and P_{Di} in terms of the "physical" fields $A_i,\ F_{0i}$, using the Dirac delta functions $\delta(\Phi^a)\delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D)$, contributes to the measure as

$$\left[\det g^4\det(\partial^2)\det\left(\epsilon_{ijk}\partial^i C_1^j C_2^k\right)\det\left(\epsilon_{ijk}\partial^i M_1^j M_2^k\right)\right]^{-1}.$$

Moreover, here F_{ij} is given by (2.16) and we performed the change of variables $F_{ij}\to A_i$. We choose domains of the integrals such that in (2.12) we can perform the replacement

$$DF_{ij}\delta(\epsilon^{klm}\partial_k F_{lm})\delta(C_n^i(P_{Di}+\frac{1}{2}\epsilon_{ijk}F^{jk}))\to\det(\partial^2)DA_i\delta(\partial_j A^j)\delta\left(C_n^i(P_{Di}+\epsilon_{ijk}\partial^j A^k)\right). \tag{2.19}$$

One can observe that $\det(\partial^2)$ should be included in the measure when one deals with the gauge fields A_i satisfying the condition (2.17), considering this change of variables from the beginning with an appropriate change of the momenta $P_{ij}\to P_i$ where the latter are canonical momenta of A_i.

Observe that in (2.18) the variables F_{0i} can be renamed as

$$F_{0i}=-g^2P_i, \tag{2.20}$$

where P_i are the canonical momenta associated to A_i. Thus, (2.18) becomes

$$\begin{aligned}Z=\det g^{-4}\int D\mathbf{A}D\mathbf{P}\det(\partial^2)\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})\\ \exp\left\{i\int d^3x[P_i\dot{A}^i-\frac{g^2}{2}P_iP^i-\frac{1}{4g^2}F^{ij}F_{ij}]\right\}.\end{aligned} \tag{2.21}$$

Although (2.20) is resulted after performing functional integrals in (2.12), we could derive it from the constraint structure using the Dirac brackets:

$$\{F_{0i}(x), P_{Dj}(y)\}_{\rm Dirac} = \{F_{0i}, P_{0k}\}\{P_{0k}, \Phi^4_l\}^{-1}\{\Phi^4_l, P_{Dj}\} \\ = g^2\epsilon_{ikj}\frac{\partial\delta^3(x-y)}{\partial x_k}. \tag{2.22}$$

Making use of (2.16) in $\chi^2_i = 0$ yields

$$P_{Di} = -\epsilon_{ijk}\partial^j A^k. \tag{2.23}$$

Plugging (2.23) into the left hand side of the Dirac bracket (2.22), leads to

$$-\epsilon_{jkl}\frac{\partial}{\partial y_k}\{F_{0i}(x), A_l(y)\}_{\rm Dirac} = g^2\epsilon_{ijk}\frac{\partial\delta^3(x-y)}{\partial x_k}, \tag{2.24}$$

which is solved as

$$\{F_{0i}(x), A_j(y)\}_{\rm Dirac} = g^2\delta_{ij}\delta^3(x-y). \tag{2.25}$$

Thus, (2.20) follows.

We choose the normalization such that partition function for Maxwell theory in hamiltonian formalism is given by

$$Z_H \equiv Z_N(g) = \det g^{-2}\int D\mathbf{A}D\mathbf{P}\det(\partial^2)\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A}) \\ \exp\left\{i\int d^3x[P_i\dot{A}^i - \frac{g^2}{2}P_iP^i - \frac{1}{4g^2}F^{ij}F_{ij}]\right\}. \tag{2.26}$$

We denoted the normalized partition function as $Z_N(g)$. The normalized partition function of the dual theory in phase space is

$$Z_{HD} = Z_N(g^{-1}) = \det g^2\int D\mathbf{A}D\mathbf{P}\det(\partial^2)\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A}) \\ \exp\left\{i\int d^3x\left[P_i\dot{A}^i - \frac{1}{2g^2}P_iP^i - \frac{g^2}{4}F^{ij}F_{ij}\right]\right\}, \tag{2.27}$$

where we renamed $A^i_D,\ P^i_D$ as $A^i,\ P^i$. By comparing Z obtained in (2.15) and (2.21) we conclude that in hamiltonian formalism partition functions for Maxwell theory and its dual are the same

$$Z_H = Z_{HD}, \tag{2.28}$$

which can equivalently be written in terms of the normalized partition functions as

$$Z_N(g) = Z_N(g^{-1}). \tag{2.29}$$

This result was obtained in [2] in terms of canonical transformations without gauge fixing factor and with another normalization.

3. Relations between the electric-magnetic duality and the dual actions of noncommutative $U(1)$ theory

Noncommuting coordinates are operators even at the classical level. In spite of this fact we can treat them as the usual commuting coordinates by replacing operator products with $*$–products. Utilizing the latter a noncommutative $U(1)$ gauge theory is defined which can be written in terms of the usual gauge fields, after performing the Seiberg–Witten map, as[3]

$$S_{NC} = -\frac{1}{4g^2}\int d^4x \left(F_{\mu\nu}F^{\mu\nu} + 2\theta^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu} - \frac{1}{2}\theta^{\mu\nu}F_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{3.1}$$

at the first order in the noncommutativity parameter $\theta^{\mu\nu}$ which is constant and antisymmetric. Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Dual of (3.1) is obtained in [4] as

$$S_{NCD} = -\frac{g^2}{4}\int d^4x \left(F_D^{\mu\nu}F_{D\mu\nu} + 2\tilde{\theta}^{\mu\nu}F_{D\nu\rho}F_D^{\rho\sigma}F_{D\sigma\mu} - \frac{1}{2}\tilde{\theta}^{\mu\nu}F_{D\mu\nu}F_{D\rho\sigma}F_D^{\rho\sigma}\right), \tag{3.2}$$

where $F_{D\mu\nu} = \partial_\mu A_{D\nu} - \partial_\nu A_{D\mu}$ and

$$\tilde{\theta}^{\mu\nu} = \frac{g^2}{2}\epsilon^{\mu\nu\rho\sigma}\theta_{\rho\sigma}.$$

Obviously, if the original theory (3.1) possesses spatially noncommutative coordinates, in the dual theory (3.2) time is effectively noncommuting with spatial coordinates.

We would like to discover relations between electric–magnetic duality and the (S) duality transformation for noncommutative $U(1)$ gauge theory in configuration space. Let us write the actions (3.1) and (3.2) in terms of the electric and magnetic fields: When the magnetic field vector

$$B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk} \tag{3.3}$$

and the electric field vector $E_i = F_{0i}$ are employed, the original action (3.1) becomes[9]

$$S_{NC} = \int d^4x \left[\frac{1}{2g^2}(\mathbf{E}^2 - \mathbf{B}^2)(1 - \theta\cdot\mathbf{B}) + \frac{1}{g^2}\theta\cdot\mathbf{E}\mathbf{E}\cdot\mathbf{B}\right], \tag{3.4}$$

where the vector θ is defined by $\theta^{ij} = \epsilon^{ijk}\theta_k$.

For the dual case we adopt the same notation: $E_i = F_{D0i}$ and

$$B_i = -\frac{1}{2}\epsilon_{ijk}F_D^{jk}. \tag{3.5}$$

Hence, the dual action (3.2) can be written as

$$S_{NCD} = \int d^4x \left[\frac{g^2}{2}(\mathbf{E}^2 - \mathbf{B}^2)(1 + \tilde{\theta}\cdot\mathbf{E}) - g^2\tilde{\theta}\cdot\mathbf{B}\mathbf{E}\cdot\mathbf{B}\right], \tag{3.6}$$

where $\tilde{\theta}$ vector is defined as $\tilde{\theta}^i \equiv \tilde{\theta}^{0i}$.

One can observe that under the transformation

$$\mathbf{E} \to g^2\mathbf{B}, \quad \mathbf{B} \to -g^2\mathbf{E}, \tag{3.7}$$

(3.4) is mapped into the dual action (3.6) up to an overall minus sign. This is a well known property of abelian gauge theory action. Thus, it persists in the noncommutative theory.

We also would like to obtain relations between the electric–magnetic duality and the (S) duality transformations of the noncommutative $U(1)$ theory in hamiltonian formalism. Canonical hamiltonian associated with (3.1) can be derived as

$$\begin{aligned} H_{NC} = \int d^3x \Big[& \frac{g^2}{2}P_i^2 + \frac{1}{4g^2}F_{ij}F^{ij} + \frac{1}{2g^2}\theta^{ij}F_{jk}F^{kl}F_{li} - \frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl} \\ & + g^2\theta^{ij}P_jP^kF_{ki} - \frac{g^2}{4}\theta^{ij}F_{ji}P_k^2\Big], \end{aligned} \tag{3.8}$$

where we choose the subsidiary condition $A_0 = 0$ which corresponds to the constraint $P_0 = 0$. Furthermore, there is the constraint $\partial_i P^i = 0$. Hamiltonian of the dual noncommutative $U(1)$ gauge theory (3.2) is obtained in [5] by two different approaches as[2]

$$\begin{aligned} H_{NCD} = \int \; & d^3x \Big[\frac{1}{2g^2}P_{Di}^2 + \frac{g^2}{4}F_{Dij}F_D^{ij} + \frac{1}{2g^4}\tilde{\theta}_{0i}P_D^iP_{Dj}^2 + \frac{1}{4}\tilde{\theta}_{0i}P_D^iF_{Djk}F_D^{jk} \\ & + \tilde{\theta}_{0i}F_D^{ij}F_{Djk}P_D^k\Big] \end{aligned} \tag{3.9}$$

with the constraint $\partial_i P_D^i = 0$ after setting $P_{D0} = 0, \; A_{D0} = 0$.

Let us introduce the vector field $P_i = g^{-2}D_i$ and the magnetic fields as before (3.3). Hence, we write the hamiltonian (3.8) as

$$H_{NC} = \int d^3x \left[\frac{1}{2g^2}(\mathbf{D}^2 + \mathbf{B}^2) - \frac{1}{2g^2}\theta\cdot\mathbf{B}(\mathbf{B}^2 - \mathbf{D}^2) - \frac{1}{g^2}\theta\cdot\mathbf{D}\mathbf{B}\cdot\mathbf{D}\right]. \tag{3.10}$$

Similarly, let us introduce $P_{Di} = g^2 D_i$ and the magnetic field as in (3.5). Then, the hamiltonian (3.9) becomes

$$H_{NCD} = \int d^3x \left[\frac{g^2}{2}(\mathbf{D}^2 + \mathbf{B}^2) - \frac{g^2}{2}\tilde{\theta}\cdot\mathbf{D}(\mathbf{D}^2 - \mathbf{B}^2) - g^2\tilde{\theta}\cdot\mathbf{B}\mathbf{B}\cdot\mathbf{D}\right]. \tag{3.11}$$

One can show that under the map

$$\mathbf{D} \to -g^2\mathbf{B}, \quad \mathbf{B} \to g^2\mathbf{D} \tag{3.12}$$

the hamiltonian (3.10) transforms into the dual hamiltonian (3.11). Thus, noncommutative electric-magnetic duality transformation in hamiltonian formulation is

[2]There are some misprints in [5] which are corrected here.

given by (3.12). Observe that the lagrangian and the hamiltonian description of electric–magnetic duality transformations, (3.7) and (3.12), seem to be “inverted”.

Definition of the canonical momenta P_i following from (3.1) can be used to express P_i in terms of the electric field $E_i = F_{0i}$. Then, one can express the hamiltonian (3.8) as[9]

$$H_{NC} = \int d^3x \left[\frac{1}{2g^2}(\mathbf{E}^2 + \mathbf{B}^2)(1 - \theta \cdot \mathbf{B}) + \frac{1}{g^2}\theta \cdot \mathbf{E}\mathbf{E} \cdot \mathbf{B} \right]. \tag{3.13}$$

Analogously, the canonical momenta P_{Di} derived from (3.2) can be expressed in terms of the electric field $E_i = F_{D0i}$. Making use of it in the hamiltonian (3.9) one obtains

$$H_{NCD} = \int d^3x \left[\frac{g^2}{2}(\mathbf{E}^2 + \mathbf{B}^2) + g^2\tilde{\theta} \cdot \mathbf{E}\mathbf{E}^2 \right]. \tag{3.14}$$

(3.13) and (3.14) are not related with a transformation resembling the electric–magnetic duality transformation (3.7).

Electric–magnetic duality transformation of the noncommutative hamiltonians cannot be given in terms of $\mathbf{E},\ \mathbf{B}$ fields but using $\mathbf{D},\ \mathbf{B}$. This is an expected result: Hamiltonians should be written in the momenta P_i or P_{Di} not by using the “velocities” F_{0i} or F_{D0i}. In the commuting case this difference does not appear due to the fact that $\mathbf{P} = \mathbf{E}$.

4. Partition functions for the noncommutative $U(1)$ theory and its dual

The noncommutative $U(1)$ action (3.1) and its dual (3.2) can be derived from the parent action[4]

$$S_{NP} = -\frac{1}{4g^2}\int d^4x \Big(F_{\mu\nu}F^{\mu\nu} + 2\theta^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu} - \frac{1}{2}\theta^{\mu\nu}F_{\nu\mu}F_{\rho\sigma}F^{\sigma\rho} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}A^{\mu}_{D}\partial^{\nu}F^{\rho\sigma} \Big), \tag{4.1}$$

where $F_{\mu\nu}$ are not composed of any other field. We only deal with the first order approximation in $\theta_{\mu\nu}$. To acquire hamiltonian formalism we introduce the canonical momenta $P_{\mu\nu},\ P_{D\mu}$ corresponding to the configuration space variables $F_{\mu\nu},\ A_{D\mu}$. Definitions of the canonical momenta P^{μ}_{D} and $P_{\mu\nu}$ yield the primary constraints

$$\tilde{\Phi}^1_{\mu\nu} \equiv P_{\mu\nu} \approx 0, \tag{4.2}$$

$$\tilde{\xi}^1 \equiv P_{D0} \approx 0, \tag{4.3}$$

$$\tilde{\chi}^2_i \equiv P_{Di} + \frac{1}{2}\epsilon_{ijk}F_{jk} \approx 0, \tag{4.4}$$

which are weakly vanishing. One can show that canonical hamiltonian related to (4.1) is

$$H_{NPC} = \int d^3x \Big[-\frac{1}{2}\epsilon_{ijk}\partial^i A_D^0 F^{jk} + \epsilon_{ijk}\partial^i A_D^j F^{0k} + \frac{1}{2g^2}F_{0i}F^{0i} + \frac{1}{4g^2}F_{ij}F^{ij} + \frac{1}{g^2}F^{0i}F_{ij}\theta^{jk}F_{k0} + \frac{1}{2g^2}F^{ij}F_{jk}\theta^{kl}F_{li} - \frac{1}{4g^2}\theta^{ij}F_{ij}F_{0k}F^{0k} - \frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl}\Big]. \tag{4.5}$$

Consistency of the primary constraints (4.2)–(4.4) with the hamiltonian equations of motion originating from (4.5), leads to the secondary constraints

$$\tilde{\Phi}^3 \equiv \{P_{D0}, H_{NPC}\} = \epsilon_{ijk}\partial^i F^{jk} \approx 0, \tag{4.6}$$

$$\tilde{\chi}_i^4 \equiv \{P_{0i}, H_{NPC}\} = F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i} - g^2\epsilon_{ijk}\partial^j A_D^k \approx 0. \tag{4.7}$$

Like the commuting case $\tilde{\xi}^1$ and the linear combination of (4.4) and (4.6)

$$\tilde{\xi}^2 \equiv \partial_i\tilde{\chi}_i^2 - \frac{1}{2}\tilde{\Phi}^3 = \partial_i P_D^i \approx 0 \tag{4.8}$$

are first class constraints. Curl of χ_i^2 leads to two linearly independent second class constraints:

$$\tilde{\Phi}_n^2 \equiv C_n^i\tilde{\chi}_i^2 \equiv K_n^i\epsilon_{ijk}\partial_j\tilde{\chi}_k^2 \approx 0, \tag{4.9}$$

where $n = 1, 2$. Analogous to the commuting case, instead of $\tilde{\chi}_i^4$ we deal with the following set of second class constraints

$$\tilde{\Phi}_n^4 \equiv M_n^i\tilde{\chi}_i^4 \equiv L_n^i\epsilon_{ijk}\partial_j\tilde{\chi}_k^4 \approx 0, \tag{4.10}$$

$$\tilde{\Phi}_3^4 \equiv \partial_i\left(F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \tfrac{1}{2}\theta^{jk}F_{kj}F_{0i}\right) \approx 0. \tag{4.11}$$

K_n^i and L_n^i are some constants which should be determined by taking into account the other constraints when they vanish strongly. The constraints (4.2) and (4.6) are also second class. Structure of the constraints is similar to the commuting case discussed in Section 2. In fact, the number of physical phase space fields is four.

In phase space, partition function associated with the parent action for noncommutative $U(1)$ theory (4.1) is defined as

$$\tilde{Z} = \int DPDP_DDFDA_D\;\tilde{\Delta}\exp\left\{i\int d^3x\left[P_D^\mu\dot{A}_{D\mu} + P_{\mu\nu}\dot{F}^{\mu\nu} - H_{NPC}\right]\right\}. \tag{4.12}$$

Indices of the integration variables are suppressed. We have adopted the gauge fixing conditions

$$\tilde{\Lambda}^1 = A_{D0} \approx 0, \qquad \tilde{\Lambda}^2 = \partial_i A_{Di} \approx 0, \tag{4.13}$$

for the first class constraints (4.3) and (4.8). Therefore, the measure $\tilde{\Delta}$ is

$$\tilde{\Delta} = \det\{\tilde{\xi}^\alpha, \tilde{\Lambda}^\beta\}\det^{\frac{1}{2}}\{\tilde{\Phi}^a, \tilde{\Phi}^b\} \prod_{\sigma=1}^{2} \delta(\tilde{\xi}^\sigma)\delta(\tilde{\Lambda}^\sigma) \prod_{c=1}^{4} \delta(\tilde{\Phi}^c). \tag{4.14}$$

Contribution of the first class constraints $\tilde{\xi}^\alpha$ and their subsidiary conditions $\tilde{\Lambda}^\alpha$ to the measure is

$$\det\{\tilde{\xi}^\alpha, \tilde{\Lambda}^\beta\} = \det(\partial^2). \tag{4.15}$$

The second class constraints $\tilde{\Phi}^a$ contribute to the measure as

$$\det^{\frac{1}{2}}\{\tilde{\Phi}^a, \tilde{\Phi}^b\} = \det\left(\epsilon_{ijk}\partial^i M_1^j M_2^k\right) \det\left(\epsilon_{ijk}\partial^i C_1^j C_2^k\right) \det\left(-1 + \frac{1}{2}\theta^{ij}F_{ji}\right). \tag{4.16}$$

$\epsilon_{ijk}\partial^i M_1^j M_2^k$ and $\epsilon_{ijk}\partial^i C_1^j C_2^k$ denote multiplication of three linear differential operators and as usual, determinants of them are defined as multiplication of the eigenvalues of the linear operators. The last term in (4.16) is to be interpreted as multiplication of the value of $\left(-1 + \frac{1}{2}\theta^{ij}F_{ji}\right)$ over all space–time. The determinants should be regularized, however as we will show, our results are independent of their regularizations.

Performing functional integrations over $F^{\mu\nu}$ and $P_{\mu\nu}$ in (4.12) we obtain

$$\begin{aligned}\tilde{Z} = &\int D\mathbf{A}_D D\mathbf{P}_D \delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D)\det(\partial^2) \\ &\exp\left\{i\int d^3x\Big[P_{Di}\dot{A}_D^i - \frac{1}{2g^2}P_{Di}P_D^i - \frac{g^2}{4}F_{Dij}F_D^{ij} \right.\\ &\left.+\frac{1}{2g^4}\tilde{\theta}^{0i}P_{Di}P_D^2 + \tilde{\theta}^{0i}F_{Dij}F_D^{jk}P_{Dk} + \frac{1}{4}\tilde{\theta}^{0i}P_{Di}F_D^2\Big]\right\}.\end{aligned} \tag{4.17}$$

The determinant (4.16) is canceled[3] when we used $\delta(\tilde{\Phi}^a)$ to express the "redundant" fields $F^{\mu\nu}$, $P_{\mu\nu}$ in terms of the "physical" fields A_D^i, P_D^i. Obviously, there are other solutions of (4.10) and (4.11) which would be useful to express another set of fields in terms of the remaining ones. We take the solution yielding the partition function which we desire. We observe that in (4.17) the exponential term is the first order action of the dual noncommutative $U(1)$ theory whose hamiltonian is (3.9).

Like the commuting case discussed in Section 2, when $\tilde{\Phi}^3 = 0$ is used to write

$$F_{ij} = \partial_i A_j - \partial_j A_i,$$

we demand that the constraint

$$\partial_i A^i \approx 0$$

[3]Obviously, to obtain (4.17) one does not need to separate $\tilde{\chi}_i^4$ as (4.10)– (4.11).

should be fulfilled. Moreover, when we change the variables $F_{ij} \to A_i$ we choose the domains of integrations in (4.12) such that (2.19) is satisfied. Equipped with these, we perform integrations over the fields $A_{D\mu},\ P_{D\mu},\ P_{\mu\nu}$ in (4.12) which yield

$$\begin{aligned}\tilde{Z} = \det g^{-4} \int & D\mathbf{A} DF^{0i} \delta(\partial \cdot \mathbf{A}) \det(\partial^2) \det\left(-1 + \frac{1}{2}\theta^{ij} F_{ji}\right) \\ & \delta\left(\partial_i (F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i})\right) \\ & \exp\left\{ i \int d^3x \Big[\frac{1}{g^2} \dot{A}^i (F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i}) \right. \\ & + \frac{1}{2g^2} F_{0i}F^{0i} - \frac{1}{4g^2}F_{ij}F^{ij} + \frac{1}{g^2}F^{0i}F^{0j}F_{jk}\theta^{ki} - \frac{1}{4g^2}\theta^{jk}F_{jk}F_{0i}F^{0i} \\ & \left. + \frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl} \Big] \right\}. \end{aligned} \tag{4.18}$$

We made use of the fact that employing $\delta(\tilde{\Phi}^a)\delta(\partial \cdot \mathbf{P}_D)\delta(\partial \cdot \mathbf{A}_D)$ to express $P_D^i,\ A_D^i$ in terms of F_{0i} and A_i gives the following contribution to the measure

$$\left[\det g^4 \det(\partial^2) \det\left(\epsilon_{ijk}\partial^i C_1^j C_2^k\right) \det\left(\epsilon_{ijk}\partial^i M_1^j M_2^k\right)\right]^{-1}.$$

To deal with P_i which are the canonical momenta of A_i, let us adopt the change of variables,

$$g^2 P^i = F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i}, \tag{4.19}$$

by inspecting the terms multiplying $\dot{A}^i$. Thus, the partition function (4.18) is written as

$$\begin{aligned}\tilde{Z} = \det g^{-4} \int & D\mathbf{A} D\mathbf{P} \delta(\partial \cdot \mathbf{P}) \delta(\partial \cdot \mathbf{A}) \det(\partial^2) \\ & \exp\left\{ i \int d^3x \Big[\dot{A}^i P_i - \frac{g^2}{2}P_i P^i - \frac{1}{4g^2}F_{ij}F^{ij} - g^2\theta^{ij}P_i P^k F_{jk} \right. \\ & \left. + \frac{g^2}{4}\theta^{ij}F_{ji}P^2 + \frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl} \Big] \right\}. \end{aligned} \tag{4.20}$$

In the exponential factor of (4.20) we recognize the hamiltonian of the noncommutative $U(1)$ theory (3.8).

It could be possible to show that the canonical momenta P_i are given as in (4.19) using the Dirac brackets:

$$\begin{aligned}\{F_{0i}(x), P_{Dj}(y)\}_{\mathrm{Dirac}} = \{F_{0i}, P_{0k}\}\{P_{0k}, \tilde{\Phi}_l^4\}^{-1}\{\tilde{\Phi}_l^4, P_{Dj}\} = g^2\epsilon_{jkl}[\delta_i^k + F^{km}\theta_{mi} \\ + F_{im}\theta^{mk} + \frac{1}{2}\delta_i^k\theta^{mn}F_{nm}]\partial_x^l \delta^3(x-y). \end{aligned} \tag{4.21}$$

Vanishing of (4.4) and (4.6) strongly, permit us to write the left hand side of (4.21) equivalently as

$$\{F_{0i}(x), P_{Dj}(y)\}_{\mathrm{Dirac}} = -\epsilon_{jkl}\partial_y^k \{F_{0i}(x), A^l(y)\}_{\mathrm{Dirac}}. \tag{4.22}$$

By comparing the right hand sides of (4.21) and (4.22) we observe that they are compatible when

$$F_{0i} = -g^2(P_i + F_{ij}\theta^{jk}P_k + F^{jk}\theta_{ki}P_j - \frac{1}{2}F_{jk}\theta^{kj}P_i). \tag{4.23}$$

Solving this equation for P_i at the first order in θ_{ij} gives rise to (4.19).

We adopt the normalization consistent with Section 2, to write partition function of the noncommutative $U(1)$ theory in phase space as

$$\begin{aligned} Z_{NC} = \det g^{-2} \int & D\mathbf{A}D\mathbf{P}\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})\det(\partial^2) \\ & \exp\Big\{i\int d^3x\Big[\dot{A}^iP_i - \frac{g^2}{2}P_iP^i - \frac{1}{4g^2}F_{ij}F^{ij} - g^2\theta^{ij}P_iP^kF_{jk} \\ & +\frac{g^2}{4}\theta^{ij}F_{ji}P^2 + \frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl}\Big]\Big\}. \end{aligned} \tag{4.24}$$

Accordingly, the dual partition function is given by

$$\begin{aligned} Z_{NCD} = \det g^{2} \int & D\mathbf{A}D\mathbf{P}\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})\det(\partial^2) \\ & \exp\Big\{i\int d^3x\Big[P_i\dot{A}^i - \frac{1}{2g^2}P_iP^i - \frac{g^2}{4}F_{ij}F^{ij} \\ & +\frac{1}{2g^4}\tilde{\theta}^{0i}P_iP^2 + \tilde{\theta}^{0i}F_{ij}F^{jk}P_k + \frac{1}{4}\tilde{\theta}^{0i}P_iF^2\Big]\Big\}, \end{aligned} \tag{4.25}$$

where we renamed $A_{Di},\ P_{Di}$ as $A_i,\ P_i$.

We conclude that in phase space, partition functions for the noncommutative $U(1)$ theory and its dual are the same

$$Z_{NC} = Z_{NCD}. \tag{4.26}$$

This result demonstrates that strong–weak duality transformation is helpful to make calculations in weak coupling regions to extract information about physical quantities in the strong coupling regions.

We would like to emphasize the difference between the results obtained for the commutative case (2.29) and for the noncommutative $U(1)$ theory (4.26). In $U(1)$ gauge theory, partition functions for the initial and the dual theories, (2.26) and (2.27), are equivalent and they are related with the map $g\to g^{-1}$. However, the partition function of noncommutative $U(1)$ (4.24) does not yield the partition function of its dual (4.25) by only inverting the coupling constant, although they are equivalent.

Application of the approach presented here to noncommutative supersymmetric $U(1)$ gauge theory whose parent actions were studied in [10], may shed light on the duality symmetry of the supersymmetric noncommutative theory.

We dealt with free theories, although introducing source terms into the starting path integral (4.12) to gain insight about relations of the Green functions of the noncommutative $U(1)$ theory and its dual is very important.

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